

# Translator's Afterword

## *Three Controversies about*

## *Mathematics, Geometry, and Education*

What we think of mathematics, and how we teach and learn it (or not), determines to a large degree the place it takes in our culture. Regardless of what we think, mathematics enters our life by providing us with idealized models of real phenomena and showing us how to deal with them logically and creatively. At the end of a traditional course in elementary geometry, a subject seen for centuries as the essence of mathematics, it is tempting to examine whether what we think of it is true. Here is a brief summary of the three (most influential in my opinion) *common views* of mathematics, and of geometry in particular:

\* *Mathematics is a relative wisdom; mathematical theorems, being logical consequences of axioms, are representative of real world relationships only to the degree that the axioms are.*

\*\* *A key virtue of mathematics (as well as the notorious difficulty of it) resides in the strict deductive nature of mathematical reasoning, as is best demonstrated by elementary Euclidean geometry.*

\*\* *To offset the difficulty and provide for success in education, early exposure to elements of Euclidean geometry is highly recommended.*

Usually such views are conveyed to the broad educated audience via the high-school geometry course, but they sound self-explanatory and uncontroversial anyway, and are readily endorsed by those who are professionally affiliated with mathematical education.

In these notes, we will see, drawing some examples from the main text of this book, that these views are essentially *misleading*, as they are either outdated or a result of terminological confusion and mis-information about the history and essence of mathematics, and that the direction in education suggested by them is rather perverted.

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It is true that classical elementary geometry was developed by postulating basic properties of space in the form of *axioms*, and logically deriving further properties from them. It is also true that modern mathematics often relies on the *axiomatic method*. It turns out however that what is meant by *axioms* has changed. Nowadays, axioms are used for unification purposes, i.e. in order to study several similar examples at once. For instance (§140), *symmetricity* and *bilinearity* are axioms defining an *inner product*, a notion

that unifies the *Euclidean* (§140) and *Minkowski* (§145) dot products. Another example: the eight axioms of a *vector space* (§137) unify coordinate vectors (§142) with geometric ones (§119).

Studying properties of several similar objects at once is a very common method, and one sure way by which mathematics *saves effort* (using an expression of Ron Aharoni [15]) and thus becomes useful. Axioms here are simply part of a definition, i.e. a *convention* which calls an object by such-and-such a name if it possesses the required properties; in mathematics they are not “self-evident truths accepted without proof,” as the conventional wisdom would have it. For instance, the definition of *regular polyhedra* in §84 is *axiomatic*, as opposed to the *constructive* description of the five Platonic solids given in §§85–86. The theorem of §87 illustrates the use of the axiomatic method for purposes of *classification* (of regular polyhedra, in this example). Similar applications are found in §142 (“uniqueness” of Euclidean geometry) and §§149–151 (characterization of isometries).

It often happens that general results and concepts of mathematics, initially motivated by known examples, are successfully applied in unexpected ways to new situations. The alternative scenario: an axiomatic theory developed with *no* examples known to satisfy the axioms, is rather unusual. Thus mathematics appears today not as a “relative wisdom” (where conclusions hold *if* the axioms are satisfied) but as a *science* motivated by studying important and interesting examples, for which the conclusions *do* hold because the axioms *are* satisfied. Such examples often come as mathematical models of real phenomena. The most basic of these models deal with comparing finite sets of objects, and the correct way of manipulating them is not decided by any system of axioms. It is learned (even by advanced mathematicians) through the tedious process of *counting* — in childhood.

Then what about classical Euclidean geometry? The way it was developed seems today quite similar to some advanced branches of theoretical physics, notably *string theory*. Sometimes physics goes beyond of what is known in mathematics and needs mathematical models that are not available. Then physicists use *heuristic* methods: they *postulate* the existence of certain models with certain properties, and prescribe certain rules of manipulating them, even though there is not a single example that fits the description. This is similar to how *non-Euclidean geometry* first emerged in the work of Lobachevsky and Bolyai (§134). Later, if physicists' expectations turn out to be reasonable, mathematicians construct the required models, such as the hyperbolic (§146) and projective (§144) planes in the case of non-Euclidean geometry. But until then, heuristic methods prevail in describing physical reality.

This is how *space* is described in classical Euclidean geometry, both in antiquity (see §133) and in modern age (Book I, §§1–5). One would examine the images of a stretched thread, a light ray, or the surface of a pond or desk, introduce *infinitesimally* thin and *infinitely* spread idealizations of these objects, postulate those properties of lines and planes which appear obvious from the heuristic point of view, and then obtain further properties

by reasoning. The reader can check that this was the way the foundations of geometry were treated by Euclid [1]. The physicists' heuristic approach to the foundations of elementary geometry worked well for mankind for over two millennia. This approach should suffice even today for anyone studying the subject for the first time.

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According to the author of a modern Russian textbook [9], “Geometry is a subject for those who like to dream, draw and examine pictures, and who are good at making observations and drawing conclusions.” According to an expert at a homeschool math blog popular in the U.S., “high school geometry with its formal (two-column) proofs is considered hard and detached from practical life.” Sounds different? How come?

This time, it is *geometry* that means two different things. In various countries, at different periods, the same new current in math education emerged. The main idea was to bring high-school mathematics to a level contemporary to the 20th century. In geometry, it meant introducing set-theoretic terminology and emphasizing the role of geometric transformations. In Russia, *Kolmogorov's reform* took place in late seventies, and was immediately recognized as a failure (which seems to be the fate — for a variety of causes — of all reforms in education). It did affect the quality of instruction, but it shook only slightly the status of geometry as the most inspiring part of the math curriculum. The analogous reform in the U.S., which took place in the sixties and was dubbed *New Math*, was accompanied also with the intention of introducing mathematics “the correct way” right from the start (as opposed to raising the level of abstraction in stages). For geometry this meant: to erect it on a rigorous axiomatic foundation.

The search for a solid foundation for geometry has played an important role in the development of mathematics (see §134). As was mentioned in the previous section, this problem emerges not in a first study of the subject, but later, when the building is already there and the question of what it stands on remains. Modern mathematics solves this problem by introducing geometry through vector algebra (as it is done in §§136–141). The vector approach is considered “the royal road to geometry”: it is logically simple, and intuitively transparent, since vectors come from physics. It also brings into elementary geometry new problems and methods (see §§119–132), and paves the road to more advanced mathematics, such as *linear algebra*.

The New Math reform attempted to bring rigor into a beginner's course of elementary geometry by following, albeit loosely, Hilbert's axiomatic approach (§135). Apparently, Hilbert's monograph [2] was misconstrued as a contemporary exposition of elementary geometry. In fact this work played a key role in forming another branch of mathematics, *mathematical logic*, but added little to classical geometry and nothing to modern. Moreover, according to a leading French mathematician David Ruelle [12], “Hilbert's version of Euclidean geometry without the help of (1) [visual experience and intuition] and (2) [drawings] shows how hard the subject really is.”

The focus of the post-New Math geometry courses falls, therefore, on deductive reasoning, understood as the task of meticulous conversion of hy-

potheses into conclusions. The format of *two-column proofs* is implemented to streamline the process (see an example in §137): the left column is for *what* is claimed, and the right for *why*. In the genre of two-column proofs, it takes several lines to fully justify even an obvious statement (e.g. that if one angle formed by two intersecting lines is right then the other three angles they form are also right). Instead of shortcutting to deep and beautiful geometric results, these textbooks either cast these results away or render them in fine print, and dedicate whole chapters to *formal proofs* of *trivial*, i.e. relatively obvious, facts.

In real mathematics, ancient or modern, there is no such thing as “two-column” proofs (as opposed to “paragraph” ones), just as there is no division of proofs into “formal” and “informal.” What, indeed, is a proof? In science, we want to know not only *what* is true but also *why* it is so, and a proof is an answer to the latter question. There is a subtlety though.

In mathematics, we systematically use the advantage of building new knowledge upon previously established facts (and this is yet another way that mathematics saves effort). It is not prohibited even in math to use heuristic, plausible reasoning. For instance, one can form many composite numbers by multiplying a few primes, and so it seems *plausible* that prime numbers should occur sparsely among all whole numbers. While there exist mathematical theorems that make this intuition precise, the statement taken too literally is expected to be false: according to the famous *twin prime conjecture*, there are infinitely many pairs of primes that are only 2 units apart, like 29 and 31, or 41 and 43. Clearly, deriving logical conclusions from observations that are only roughly correct and admit exceptions, may lead to false results and contradictions. What is even worse, according to the rules of logic, a proposition “*A implies B*” is true when *A* is false. Hence, a single contradiction would rob one of the very means to obtain reliable conclusions by logic: if *some A* were both true and false, then so would *every B*! The method of building towers of new conclusions upon previously established facts requires, therefore, that mathematical propositions be stated in a form that would *allow no exceptions whatsoever*. Thus, the answer to the question of *why* such a proposition is true should also explain why it allows no exceptions whatsoever. Whenever an argument is neat enough to be convincing in this regard, it qualifies as a mathematical proof.

Those who manage to evade the burden of two-column proofs and succeed in studying elementary geometry know firsthand that mathematics can be valuable or difficult not due to the neat reasoning involved (which does come in handy at times), but because mathematical gems reveal themselves only when insight and ingenuity come into play.

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To repair the damage made by the increasingly formal style and shallow content of high-school geometry courses, two remedies were invented.

The first one (apparently implemented in most U.S. high-schools) was to abandon the whole subject of classical elementary geometry in favor

of elements of analytical geometry and coordinate linear algebra. This approach to geometry (see §§209–212 of Book I and §§142, 148 of Book II) is well suited for developing routine exercises and algorithmic techniques. A typical result for geometry instruction of this type is the ultimate loss of the features, such as challenge and originality, that mark good science.

In the other approach, one intends to keep elementary geometry in school (however formal and shallow, or even if only as an *honors* course) by offering a preliminary, preparatory course of *informal* geometry (as opposed to *rigorous* one). While in some cases this becomes simply the return to a traditional geometry course (similar to Book I), more often this means: rendering math by examples, and “without proof,” i.e. dogmatically. As a variation, some popular textbooks of rigorous geometry realize the same idea by exposing the reader to “formal proofs” only *after* introducing many geometric facts in a series of chapters written “informally.” Both variations fit a more general philosophy, according to which a high level of intellectual maturity is required to succeed in studying classical Euclidean geometry, and to reach this level, gradual exposure to geometric ideas is proposed. Many modern math curricula adopt this philosophy and dedicate to geometry substantial portions of study time in middle and even elementary school. As we noted earlier, these ideas sound quite reasonable, so it is worth taking a look at where they lead.

It is important to realize that mathematics per se (as opposed to the way it is taught) is not inherently evil, and so if it avoids using some simple methods, there usually are reasons for this. For example, it is not hard to measure the sum of the angles of a triangle and find that it is about  $180^\circ$ . What is not possible to do by such measuring is to figure out why all triangles have the same sum of the angles, for one thing, because there are infinitely many triangles, and for another, because that is actually false (see §144) for triangles on the surface of the globe. Approaching geometry informally (i.e. neglecting logical relations) makes it hard to determine what is true and why. In geometry education, this usually leads to the dogmatic style, and (what is even worse) mathematical *knowledge* being systematically replaced with *tautology*. To illustrate the latter point, we discuss here three exercises taken from the chapter *Geometric figures* in a popular pre-algebra textbook [13].

(1) *Classify each given triangle by its (given) angles.* To “classify” means to decide if the triangle is acute, right or obtuse. One should realize that triangles are not inherently divided into acute, right or obtuse, but it is people who *agreed* to classify triangles this way. They did so in order to express geometric knowledge, e.g. to answer the question: *Does the orthocenter of a given triangle lie inside or outside it?* The answer is *inside* for acute and *outside* for obtuse triangles. But the mere question about classifying the triangles by angles is *tautological*, as an answer would contain no geometric information beyond what is directly given.

(2) *Find the measure of each angle of a regular pentagon, given that the sum of the measures of the angles of a pentagon is  $540^\circ$ .* A totally blind space alien who has no idea what polygons, angles or degrees are,

will successfully answer this question if told that *by the very definition* a regular pentagon has five angles of equal measure:  $540^\circ$  divided by 5 is equal to  $108^\circ$ . Not only does this exercise require no information beyond a definition, but it does not even require any visual interpretation of the definition. The same answer would involve non-tautological reasoning, if the sum of the angles were not given.

(3) *Find the perimeter of each polygon* (with the lengths of the sides labeled on a diagram). The *perimeter*, defined as the distance around a figure, is a favorite geometry topic of many elementary school curricula. In fact this definition is merely an English translation of the Greek word *perimeter*. A kindergartener, asked to find the length of the fence around a lot with five sides of 1, 2, 3, 4, and 10 yards long, will be able to answer:  $1+2+3+4+10 = 20$  yards. Thus the difficulty of the whole topic is purely linguistic, namely in the use of a foreign word. To emphasize that solving such exercises is void of any geometric content, I chose unrealistic numbers: the pentagon, whose perimeter of 20 yards has just been successfully computed, cannot exist because of the triangle inequality (Book I, §49).

Of course, conventions such as definitions and notations are present in every mathematical text, since they are needed for expressing mathematical knowledge. Unfortunately, geometric portions of typical elementary school curricula are dedicated entirely to conventions and tautologies. This is not just a result of poor realization of good intentions, since it comes framed as a certain ideology. Known as *the van Hiele model*, this ideology merits a brief description.

According to the van Hiele model, the ability of a learner to process geometric knowledge is determined by the *level* of geometric abstraction achieved by this learner. At level 0, one is only able to identify geometric shapes (e.g.: this is a rectangle). At level 1, one is able to attribute properties to shapes (e.g.: a rectangle has four right angles, and two diagonals of the same length). At level 2, one becomes capable of deriving relationships between the properties (e.g.: *if* the four angles of a quadrilateral are right, *then* it must be a rectangle, and *hence* its diagonals have the same length). At level 3, one is able to appreciate an entire logical theory that tracks all properties of geometric shapes back to axioms. At level 4, one can freely navigate through and compare abstract axiomatic theories (such as non-Euclidean geometries) not relying on geometric intuition. The main point of the model is that, regardless of age, a learner cannot progress to the next level until he is firmly grounded in the previous one.

In the half century since its invention, this classification of five levels has been the subject and the basis of many projects in education, and is considered a well-established classical theory. It is quite remarkable, therefore, that at a closer look the theory itself turns out to be *almost entirely a tautology*. For comparison, imagine a “theory” claiming that high-school students are divided into three categories: those who carry less than \$20 in their pockets, those who carry from \$20 to \$100, and those who carry over \$100. One can develop a field study on a school’s campus and confirm that “the theory works!” In application to van Hiele’s levels, such

a field study has been conducted, and the results reported in the book [14]. The fact that the classification into van Hiele's levels, however smart and elegant, is merely a *definition*, and so it cannot be confirmed or disproved by any experiments, seems to escape, somehow, the researchers' attention.

The part of the van Hiele model that can be true or false (and hence is capable of carrying knowledge) consists of the claims that *a learner of geometry cannot reach the next level while bypassing the previous one*. These are four essentially independent claims (about reaching levels 1,2,3, and 4). In fact the last two are true *tautologically*, simply because *many* is more than *one*. Indeed, operating with axiomatic theories (level 4) includes operating with one of them (level 3). Likewise, deriving *all* properties of geometric figures from axioms (level 3) includes deriving *some* properties from others (level 2). What remains are the assumptions that before attempting a rigorous geometry course one has to go through two preliminary stages: first becoming familiar with basic geometric shapes, and then learning to discern their mathematical properties intuitively. These assumptions are used to justify the ways geometry is presented throughout elementary and middle school, and so they are important.

A beginner's experience with geometric shapes should not be taken lightly, since it is one of two primary places where mathematics meets the real world (the other one being *counting*). All basic notions of geometry are somehow abstracted from this experience. The trouble is that the experience is often confused with the skill of naming shapes correctly: "this is a triangle, and this is a square." Educational psychologists illustrate a typical "difficulty" with this example: a beginner would not recognize a *square* as a (special case of) *rectangle*, but would classify it as a distinct shape. In fact the beginner is right: a square is a special case of rectangle not *intrinsically*, but only *by convention*, while by another convention (see Book I, §96) a parallelogram is *not* considered a special case of trapezoid. A convention is not something one can *figure out*. In mathematics, giving names to objects is the function of *definitions*, not *theorems*. Likewise, in real life, focusing on how things are called is void of any knowledge about them, and is in this sense *meaningless*. Here are some examples of *meaningful* questions.

1. *Why are doors and windows rectangular and not triangular?* (To understand why, imagine how a triangular board with hinges would open.) This question focuses on the properties of objects as determined by their shapes, whatever the names might be.

2. *Why are sewer hole covers often shaped as disks but rarely as squares?* The conventional answer to this question says that a square, turned sideways in space, can fall into the hole it covers, but a disk cannot. This may bring up another question: *Are disks the only shapes with this property?*

3. *How would a car move if the wheels were shaped as (regular) pentagons, or hexagons?* Well, it would not move very smoothly. The wheels are mounted to the car's axes by their centers, and what matters is that the distance from the center to boundary points of the pentagon (or hexagon) varies. This question leads directly to the definition of a circle as the locus of points on the plane *equidistant* from the center.

4. A traditional technique of relocating buildings consists in placing round wood trunks of the same diameter under a (raised) house and rolling it to a new place. *Would the technique work well if the trunks had square cross sections?* In fact, what matters here is that a disk has the same *width* in every direction, and the square does not. *Are there figures of constant width other than disks?*

Generally speaking, it is not easy to invent geometry questions that are meaningful yet elementary. What helps *understanding* (as opposed to merely *naming*) geometric shapes is not classroom discussions but the fact that shapes around us do matter. One learns what a right angle is by fitting a bookshelf and a sofa bed next to each other, and encounters parallelograms and trapezoids by drawing buildings according to the rules of perspective.

Finally, let us return to the idea that an informal approach to geometry must precede the rigorous one. On the one hand, the statement sounds self-defying. If one cannot begin with the rigorous approach, then, since this is a relatively new pedagogical theory, how did people manage to learn Euclidean geometry in the previous two millennia? On the other hand, it seems obvious indeed, that Euclidean geometry is demanding of the learner's intellectual maturity, including the ability to concentrate, think, reason, meet a challenge, read a book focusing on every detail, use concise expression and precise terminology, etc.

The solution to this dilemma is very simple. The subject of Euclidean geometry does not lend itself to purely intuitive, non-rigorous treatment. It begins where Euclid began: from describing basic properties of abstract points, lines, planes, and using imagination and logic in order to discover and prove properties of geometric figures. To prepare oneself to study geometry, anything that requires imagination and logic, *apart from geometry itself*, is suitable. Mathematics of the elementary school becomes one such area, if studied not dogmatically but with full understanding of *why* it works. Meaningful geometric content is very limited there, but in basic arithmetic, one needs to go through many deep and subtle mathematical ideas in order to fully appreciate the decimal number system, standard algorithms, and operations with fractions (see [15]). To mention more: natural sciences (e.g. the structure of electron shells in atoms, the periodic table of chemical elements and genetics); computers and programming languages (e.g. the robotic system *LEGO Mindstorm*); the grammar of natural languages; music and the theory of harmony; visual arts (e.g. origami); games and puzzles (e.g. chess or the Rubik's Cube). Anything real, which is not a tautology but is rich with genuine, deep, non-trivial knowledge and structure, prepares one for studying geometry and more advanced mathematics. Everything fake: a substitute invented to facilitate instruction (be it *Informal Geometry* or even *Calculus*), has the opposite effect.





# Bibliography

- [1] Euclid. *The Thirteen Books of the Elements*. Translated with introduction and commentary by T. L. Heath. Vol. 1 (Books I–II), vol. 2 (Books III–IX), vol. 3 (Books X–XIII): Dover, New York, 1956.
- [2] David Hilbert, *The Foundations of Geometry*. Authorized translation by E. J. Townsend. The Open Court, La Salle, Illinois, 1950.
- [3] Reviel Netz. *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History*. Cambridge University Press, 1999.
- [4] A. P. Kiselev. *Kiselev's Geometry. Book I, Planimetry*. Adapted from Russian by Alexander Givental. Sumizdat, El Cerrito, 2006.

## **Editions of Kiselev's *Geometry* the translation is based on:**

- [5] A. P. Kiselev. *Elementary Geometry, for Secondary Educational Institutions*. Part I: *Planimetry*. Part II. *Stereometry*. First edition: Dumnov's Bookstore, Moscow, 1892.
- [6] A. P. Kiselev. *Elementary Geometry*. Prosveshchenie, Moscow, 1980, 1998.

## **Some ideas for additional exercises were borrowed from:**

- [7] V. B. Lidsky et al. *Problems in Elementary Mathematics*. Nauka, Moscow, 1967 (in Russian).
- [8] V. V. Prasolov. *Problems on planimetry*. Part 2. Nauka, Moscow, 1991 (in Russian).
- [9] I. F. Sharygin. *Geometry. Grades 10–11*. Drofa, Moscow, 2007; *Geometry. Stereometry. Problems*. Drofa, Moscow, 2000; *Geometry. Planimetry. Problems. Grades 9–11*. Drofa, Moscow, 2001 (in Russian).
- [10] I. H. Sivashinsky. *Problems in Elementary Mathematics*. Nauka, Moscow, 1966 (in Russian).
- [11] M. I. Skanavi (editor) *Collected Mathematics Problems for Technical College Applicants*. Vysshaya Shkola, Moscow, 1988 (in Russian).

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**Sources quoted in the Afterword:**

- [12] David Ruelle. *Conversations on mathematics with a visitor from outer space*. <http://www.ihes.fr/~ruelle/PUBLICATIONS/126imu.ps>.
- [13] M. Dolciani, R. Sorgenfrey, J. Graham. *Pre-algebra. An accelerated course*. Houghton–Mifflin Co., Boston, 1989.
- [14] D. Fuys, D. Geddes, R. Tischler. *The van Hiele model of thinking in geometry among adolescents*. Journal for Research in Mathematics Education. Monograph No. 3, NCTM, 1988.
- [15] R. Aharoni. *Arithmetic for parents. A book for grownups about children's mathematics*. Sumizdat, El Cerrito, 2007.